Modulus, Argument, Polar Form, Argand diagram and deMoivre's Theorem

1. Find the modulus and argument of

(i) $\dfrac{1+i}{1-i}$ **(ii)** $\dfrac{1+\sqrt{2}+i}{1-i}$ **(iii)** $\cos\theta$ - $i\sin\theta$ **(iv)** $1 + i\tan\theta$

In (iii) and (iv), $0<\theta<\dfrac{\pi}{2}$.

2. Show that :

(i) $|z|^2 = (\mathbf{R}(z))^2 + (\mathbf{I}(z))^2$ **(ii)** $|z| \geq |\mathbf{R}(z)| \geq \mathbf{R}(z)$ **(iii)** $|\overline{z}| = |z|$

(iv) $|z_1z_2| = |z_1|\,|z_2|$ **(v)** $\overline{z_1z_2} = \overline{z}_1\overline{z}_2$ **(vi)** $|z|\sqrt{2} \geq |\mathbf{R}(z)| + |\mathbf{I}(z)|$

(vii) $z_1\overline{z}_2 + \overline{z}_1z_2 = 2\mathbf{R}(z_1\overline{z}_2)$ **(viii)** $\left|\dfrac{z_1}{z_2+z_3}\right| \leq \dfrac{|z_1|}{||z_2|-|z_3||}$

3. If $|z-2-i| < 2$ and $|w-5-5i| < 1$, find the maximum and minimum of $|z-w|$.

4. If $w = \dfrac{z_1+z_2}{z_1-z_2}$, the numbers being complex and $z_1 \neq z_2$, show that the necessary and sufficient condition for the real part of w to be zero is $|z_1| = |z_2|$.

5. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, where $z = r(\cos\theta + i\sin\theta)$ and each a_k is real. Show that

$$|f(z)|^2 = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \cos(k-j)\theta \,.$$

6. **(i)** Given that $z_1z_2 \neq 0$, use the polar form to prove that $\mathbf{R}(z_1\overline{z}_2) = |z_1||z_2|$ if and only if

$\arg z_2 = \arg z_1 \pm 2n\pi$ ($n = 0, 1, 2, \dots$)

(ii) Given that $z_1z_2 \neq 0$, use the above result to prove that $|z_1 + z_2| = |z_1| + |z_2|$ if and only if

$\arg z_2 = \arg z_1 \pm 2n\pi$ ($n = 0, 1, 2, \dots$)

Also, note the geometric verification of this statement.

7. Describe the following loci in the Argand diagram:

(i) $\arg\dfrac{z-z_1}{z-z_2} = \dfrac{\pi}{6}$ **(ii)** $|z-z_1| - |z-z_2| = 3$

(iii) $|z+3i|^2 - |z-3i|^2 = 12$ **(vi)** $|z+3i|^2 + |z-3i|^2 = 90$.

8. Let z_0 be a fixed complex number and R a positive constant. Show why point z lies on a circle of radius R with center at $-z_0$ when z satisfies any one of the equations.

(i) $|z + z_0| = R$;

(ii) $z + z_0 = R(\cos\phi + i\sin\phi)$ where ϕ is real ;

(iii) $z\overline{z} + \overline{z}_0z + z_0\overline{z} + z_0\overline{z}_0 = R^2$

9. **(i)** Sketch on an Argand diagram the locus represented by the equation $|z - 1| = 1$.
Shade on your diagram the region for which $|z - 1| < 1$ and $\pi/6 < \arg z < \pi/3$.

(ii) Draw the line $|z| = |z - 4|$ and the half-line $\arg (z - i) = \pi/4$ in the Argand diagram.
Hence find the complex number that satisfies both equations.

10. Use the polar form to show that

(i) $i\left(1 - i\sqrt{3}\right)\left(\sqrt{3} + i\right) = 2 + 2i\sqrt{3}$

(ii) $(-1 + i)^7 = -8\,(1 + i)$

(iii) $\left(1 + i\sqrt{3}\right)^{-10} = 2^{-11}\left(-1 + i\sqrt{3}\right)$

11. Express $\sqrt{3} - i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$.

Hence show that, when n is a positive integer, $\left(\sqrt{3} - i\right)^n + \left(\sqrt{3} + i\right)^n = 2^{n+1}\cos\dfrac{n\pi}{6}$.

12. If $\left(1 + i\sqrt{3}\right)^n = a_n + ib_n$, where a_n , b_n are real numbers, show that

$$a_{n-1}b_n - a_nb_{n-1} = 4^{n-1}\sqrt{3} \qquad \text{and} \qquad a_na_{n-1} + b_nb_{n-1} = 4^{n-1} \;.$$

13. If n is a positive integer, show that

(i) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

(ii) $(1 - i\tan\theta)^n\,(1 + i\tan n\theta) = (1 + i\tan\theta)^n\,(1 - i\tan n\theta)$

(iii) $(1 + i)^{2n} + (1 - i)^{2n} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2^{n+1} & \text{if } n/2 \text{ is an even integer,} \\ -2^{n+1} & \text{if } n/2 \text{ is an odd integer.} \end{cases}$

14. If n is a positive integer, prove that $\left(\dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$.

15. Solve the equation : $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)\ \ldots\ (\cos n\theta + i\sin n\theta) = 1$.

16. If α and β are the roots of $t^2 - 2t + 2 = 0$, express α and β in the form $r(\cos\theta + i\sin\theta)$ and show $\alpha^{4m} + \beta^{4m} = (-1)^m\,2^{2m+1}$, where m is an integer.

17. a , c are positive real numbers and b is a complex number. Let $f(z) = az\bar{z} + bz + \bar{b}\bar{z} + c$ for every complex number z, where $\bar{z}$ denotes the conjugate of z . Prove the following:

(i) $af(z) = |az + \bar{b}|^2 + ac - |b|^2$

(ii) $f(z) \ge 0$ for all z if and only if $|b|^2 \le ac$

(iii) The equation $f(z) = 0$ has a solution if and only if $|b|^2 \ge ac$

18. **(i)** Prove algebraically that $|z_1 + z_2| \le |z_1| + |z_2|$ where z_1 , z_2 are complex numbers.

(ii) Show that if $|a_n| < 2$ for $1 \le n \le N$ then the equation $1 + a_1 z + \ldots + a_N z^N = 0$ has no solution such that $|z| < \frac{1}{3}$.

19. By considering the modulus of the left-hand side, prove that all the roots of the equation

$$z^n \cos \theta_0 + z^{n-1} \cos \theta_1 + \ldots + \cos \theta_n = 2$$

where $\theta_0 , \ldots , \theta_n$ are real, lie outside the circle $|z| = \frac{1}{2}$.

20. **(i)** Prove that, for any complex numbers z_1 , z_2 , $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

(ii) Two sequences $a_0 , a_1 , a_2 , \ldots$ and $b_0 , b_1, b_2 ,\ldots$ of complex numbers are defined as follows

$$a_0 = b_0 = c = \cos\theta + i \sin\theta$$

and $a_{k+1} = a_k + c^{2^k} b_k ,$ $b_{k+1} = a_k - c^{2^k} b_k$, for $k \ge 0$.

Show that $|a_n|^2 + |b_n|^2 = 2^{n+1}$ for all integers $n \ge 0$.

Hence show that $|a_n| \le \left(\sqrt{2}\right)^{n+1}$ and $|b_n| \le \left(\sqrt{2}\right)^{n+1}$.

21. **(i)** Prove that, if z's are any complex numbers and c is positive, then

$$|z_1 + z_2|^2 \le (1 + c) |z_1|^2 + (1 + c^{-1}) |z_2|^2 .$$

Under what condition does the sign of equality hold ?

(ii) Prove also that, if the a's are positive numbers such that $a_1^{-1} + \ldots + a_n^{-1} = 1$, then $|z_1 + \ldots + z_n|^2 \le a_1 |z_1|^2 + \ldots + a_n |z_n|^2$.